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Supersymmetric Moyal–Lax representation

Ashok Das[1] **and Ziemowit Popowicz**[2]

[1] Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627-0171, USA
[2] Institute of Theoretical Physics, University of Wrocław, 50-205 Wrocław, Poland

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Abstract
The super Moyal–Lax representation and the super Moyal momentum algebra are introduced and the properties of simple and extended supersymmetric integrable models are systematically investigated. It is shown that, much like in the bosonic cases, the super Moyal–Lax equation can be interpreted as a Hamiltonian equation and can be derived from an action. Similarly, we show that the parameter of non-commutativity, in this case, is related to the central charge of the second Hamiltonian structure of the system. The super Moyal–Lax description allows us to go to the dispersionless limit of these models in a singular limit and we discuss some of the properties of such systems.

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1. Introduction

Integrable models [1], both bosonic as well as supersymmetric [2–6], have played important roles in the study of conformal field theories, strings, membranes and topological field theories. In recent years, it has become known that string (membrane) theories naturally lead to non-commutative field theories [7], where usual multiplication of functions is replaced by the star product of Groenewold [8] and Moyal [9]. It is known now that Moyal brackets can be used in soliton theory as well [10–13]. In an earlier paper [14], we constructed the Moyal–Lax representation for bosonic integrable models, using the star product of Groenewold. Such an approach has some very attractive features. For example, a Moyal–Lax equation can be given the meaning of a Hamiltonian equation and can be derived from an action. We also showed that a Moyal–Lax equation naturally leads to the appropriate Lax equation, in terms of Poisson brackets, in the dispersionless limit. The Moyal–Lax equation was also shown to lead in a simple manner to the Hamiltonian structures (at least the first two) of the dispersionless systems, which was an open problem for quite some time.

In this paper, we continue our investigation of the star product and the Moyal–Lax representation for supersymmetric integrable models. These are models defined on a superspace [15] and we adopt the supersymmetric star product [16] as well as the Moyal bracket to construct and show that a consistent Moyal–Lax representation for supersymmetric integrable systems can, in fact, be obtained. Much like the bosonic case, the supersymmetric Moyal–Lax representation can be given the meaning of a Hamiltonian equation and can be derived from an action in superspace. The dispersionless limit of supersymetric systems is problematic in general. We show that it is possible to obtain the dispersionless models from such a representation in a singular limit, which, however, is not very practical from a calculational point of view. Therefore, the alternate Lax descriptions obtained in the literature [17, 18] are still preferable. However, as yet, there is no systematic understanding of how to obtain such alternate Lax representations. The paper is organized as follows. In section 2, we describe the basic star product and the Moyal bracket generalized to superspace. We also present various other identities that are useful in deriving the Moyal–Lax representation for supersymmetric integrable systems. In section 3, we describe, in detail, the standard Moyal–Lax representation for $N = 1$ supersymmetric KdV hierarchy. We show how this equation can be thought of as a Hamiltonian equation, which can be derived from an action. We also point out how the dispersionless limit of this system can be obtained in a singular limit. In section 4, we discuss several other models with $N = 1$ supersymmetry, both in the standard and the non-standard representations, as examples (without going into too much details). In particular, we describe the non-standard representation for the supersymmetric KdV equation (sKdV), supersymmetric two-boson system (sTB), supersymmetric nonlinear Schrödinger equation (sNLS) and the supersymmetric modified KdV (smKdV) equation. In section 5, we consider the Moyal–Lax representation for systems with extended supersymmetry. We discuss the $N = 2$ sKdV systems and bring out various properties of these systems from the Moyal–Lax representation. In section 6, we present a brief conclusion. All the calculations presented in this paper have also been checked using REDUCE [19] and the package Susy2 [20].

2. Basic relations

In a bosonic phase space, the star product of two observables is defined to be

$$A(x_i, p_i) \star B(x_i, p_i) = \mathrm{e}^{\kappa \sum_{i=1}^{n}(\partial_{x_i}\partial_{\tilde{p}_i} - \partial_{p_i}\partial_{\tilde{x}_i})} A(x_i, p_i) B(\tilde{x}_i, \tilde{p}_i)|_{\tilde{x}_i = x_i, \tilde{p}_i = p_i} \tag{1}$$

where n represents the number of coordinates. Here, in principle, the deformation parameter κ can be different along different directions. However, if we impose rotational invariance, they can all be identified. In dealing with supersymmetric systems, on the other hand, the natural phase space manifold is a graded manifold with coordinates $x_i, p_i, \theta_\alpha, p_{\theta_\alpha}$, where θ_α are the fermionic coordinate and p_{θ_α} the corresponding conjugate momenta. Here, we have taken a very general set up because we will be discussing systems with simple supersymmetry as well as ones with extended supersymmetry. The Grassmann variables satisfy anti-commutation relations. Consequently, they are nilpotent and the derivatives with respect to such variables are directional. In our discussions, we will use a left derivative for the Grassmann variables. Denoting the phase space variables collectively as $z_A = (x_i, p_i, \theta_\alpha, p_{\theta_\alpha})$, we note that the star product can be generalized to such a phase space, which is a graded manifold, as [16]

$$A(z_A) \star B(z_A) = \mathrm{e}^{\kappa\left[\sum_{i=1}^{n}(\partial_{x_i}\partial_{\tilde{p}_i} - \partial_{p_i}\partial_{\tilde{x}_i}) + \sum_{\alpha=1}^{N}(\partial_{\theta_\alpha}\partial_{\tilde{p}_{\theta_\alpha}} + \partial_{p_{\theta_\alpha}}\partial_{\tilde{\theta}_\alpha})\right]} A(z_A) B(\tilde{z}_A)|_{\tilde{z}_A = z_A} \tag{2}$$

where n represents the number of bosonic coordinates while N corresponds to the number of fermionic (Grassmann) coordinates of the manifold. The relative sign of the fermionic derivative terms, as we will see, is chosen so as to bring out the Poisson brackets, on such a graded manifold with a left derivative, correctly [21, 22]. (After all, the star product can be thought of as the exponentiation of the Poisson bracket structure.)

Let us consider, in detail, the properties of such a star product in the case of a simple graded manifold with one bosonic and one fermionic coordinate, namely, $n = 1 = N$. With the star product defined as in equation (2), it is easy to verify that

$$\begin{aligned} &x \star x = x^2 \qquad p \star p = p^2 \qquad \theta \star \theta = \theta^2 = 0 \qquad p_\theta \star p_\theta = p_\theta^2 = 0 \\ &x \star \theta = x\theta = \theta \star x \qquad x \star p_\theta = xp_\theta = p_\theta \star x \qquad \theta \star p = \theta p = p \star \theta \\ &x \star p = xp + \kappa \qquad p \star x = px - \kappa \qquad \theta \star p_\theta = \theta p_\theta - \kappa \qquad p_\theta \star \theta = p_\theta \theta - \kappa. \end{aligned} \tag{3}$$

Since, on a superspace, we can have both even and odd functions (superfields), the graded Moyal bracket of these superfields can be defined to be

$$\{A, B\}_\kappa = \frac{1}{2\kappa}\left(A \star B - (-1)^{|A||B|} B \star A\right) \tag{4}$$

where $|A|, |B|$ represent the Grassmann parity of the superfields A, B respectively. It can be easily checked that with equation (4), we obtain

$$\{x, p\}_\kappa = 1 = -\{\theta, p_\theta\}_\kappa \tag{5}$$

with all other graded Moyal brackets vanishing.

It is also easy to check from the definition in equation (4) that, in the limit of vanishing κ, they lead to the correct definitions of Poisson brackets on a graded manifold. Namely, if we assume that B, F represent respectively, a bosonic and a fermionic superfield, then, it follows from equations (2), (4) that

$$\begin{aligned} \lim_{\kappa\to 0} \{B_1, B_2\}_\kappa &= \partial_x B_1 \partial_p B_2 - \partial_p B_1 \partial_x B_2 + \partial_\theta B_1 \partial_{p_\theta} B_2 + \partial_{p_\theta} B_1 \partial_\theta B_2 = \{B_1, B_2\} \\ \lim_{\kappa\to 0} \{B_1, F_2\}_\kappa &= \partial_x B_1 \partial_p F_2 - \partial_p B_1 \partial_x F_2 + \partial_\theta B_1 \partial_{p_\theta} F_2 + \partial_{p_\theta} B_1 \partial_\theta F_2 = \{B_1, F_2\} \\ \lim_{\kappa\to 0} \{F_1, B_2\}_\kappa &= \partial_x F_1 \partial_p B_2 - \partial_p F_1 \partial_x B_2 - \partial_\theta F_1 \partial_{p_\theta} B_2 - \partial_{p_\theta} F_1 \partial_\theta B_2 = \{F_1, B_2\} \\ \lim_{\kappa\to 0} \{F_1, F_2\}_\kappa &= \partial_x F_1 \partial_p F_2 - \partial_p F_1 \partial_x F_2 - \partial_\theta F_1 \partial_{p_\theta} F_2 - \partial_{p_\theta} F_1 \partial_\theta F_2 = \{F_1, F_2\}. \end{aligned} \tag{6}$$

These are, in fact, the correct definitions of Poisson brackets on a graded manifold with a left derivative [21, 22]. It follows from this, as well as from equation (5) that in the limit $\kappa \to 0$, we have the expected canonical Poisson bracket relations

$$\{x, p\} = 1 = -\{\theta, p_\theta\} \tag{7}$$

with all others vanishing.

On a simple superspace of the kind we are considering, one can define, in addition to the usual bosonic and fermionic derivatives, a covariant derivative which transforms covariantly under supersymmetry transformations, namely,

$$D = \partial_\theta + \theta \partial_x. \tag{8}$$

Furthermore, the covariant derivative satisfies the relation that

$$D^2 = \partial_x. \tag{9}$$

We note here that we can define a fermionic quantity, from the phase space variables, as

$$\Pi = -(p_\theta + \theta \star p) = -(p_\theta + \theta p) \tag{10}$$

which would satisfy

$$\Pi \cdot \Pi = 0 \qquad \Pi \star \Pi = -2\kappa p. \tag{11}$$

It is easy to check now that, independent of the Grassmann parity of a superfield A, we have

$$\{\Pi, A\}_\kappa = (DA). \tag{12}$$

Namely, the graded Moyal bracket of Π with any superfield leads to the covariant derivative acting on the superfield for any value of κ (even in the vanishing κ limit). This is, therefore, an important concept in the study of supersymmetric integrable systems. In fact, one can think of this as the generalization of the fermionic momentum variable p_θ to one which is covariant with respect to supersymmetric transformations (translations of the Grassmann coordinates).

From the definition of the star product, it is now easy to check that, for any integer n (positive or negative)

$$p^n \star A = \sum_{m=0} \binom{n}{m} (-2\kappa)^m \left(\frac{\partial^m A}{\partial x^m}\right) \star p^{n-m} \tag{13}$$

where

$$\binom{n}{m} = \frac{n(n-1)\cdots(n-m+1)}{m!} \qquad \binom{n}{0} = 1.$$

Similarly, it can be checked that (powers of Π are defined in the star product sense)

$$\begin{aligned} \Pi^{2n} \star A &= \sum_{m=0} \binom{n}{m} (-2\kappa)^{2m} (D^{2m} A) \star \Pi^{2(n-m)} \\ \Pi^{2n+1} \star A &= \sum_{m=0} \binom{n}{m} (-2\kappa)^{2m} \big((-1)^{|A|}(D^{2m} A) \star \Pi^{2(n-m)+1} + (2\kappa)(D^{2m+1} A) \star \Pi^{2(n-m)}\big). \end{aligned} \tag{14}$$

These can be thought of as the generalizations of the super-Leibniz rules [5] to the case of the star product on a superspace.

For completeness, let us note that

$$\Pi \star p = \Pi p = p \star \Pi$$

and that, for a non-vanishing κ, we can define, from equation (11),

$$\Pi^{-1} = (-2\kappa)^{-1} \Pi \star p^{-1} = (-2\kappa)^{-1} \Pi p^{-1} = (-2\kappa)^{-1} p^{-1} \star \Pi. \tag{15}$$

This inverse, on the other hand, does not exist in the vanishing κ limit. As we will see later, this is one of the sources of difficulties in taking the dispersionless limit of supersymmetric integrable systems.

3. Supersymmetric KdV hierarchy

On the phase space of a supersymmetric system, which is a graded manifold, with a star product, we can define a Lax function which depends on the phase space coordinates as well as on dynamical variables which will be superfields (either bosonic or fermionic). In fact, for a manifestly supersymmetric description, the Lax function can depend only on p, Π as well as on superfields and covariant derivatives acting on them. Thus, for example, a Lax operator can have a form of the type

$$L_n = \sum_{m=0} \Phi_m(x, \theta) \star p^{n-m} \Pi^m \tag{16}$$

where all products are star products (for example, $\Pi^m = \Pi \star \cdots \star \Pi$ with m factors). Although such Lax operators are defined as polynomials in momenta, they inherit an operator structure through the star product and define an algebra, which we will call the super Moyal momentum algebra (sMm algebra). It is easy to check that all the properties of pseudo-differential operators on a superspace carry through, in this case, with suitable redefinitions. In particular, we note that for any two arbitrary elements A and B of the sMm algebra, the super-residue (the coefficient of the Π^{-1} term) of the super Moyal bracket satisfies

$$s\mathrm{Res}\ \{A, B\}_\kappa = (DC) \tag{17}$$

so that we can define uniquely a super-trace

$$s\,\mathrm{Tr}\ A = \int \mathrm{d}x\,\mathrm{d}\theta\ s\mathrm{Res}\,A \tag{18}$$

which will satisfy cyclicity.

For a general Lax operator of the type in equation (16), one can readily check that a super Moyal–Lax equation of the form

$$\frac{\partial L_n}{\partial t_k} = \left\{L_n, \left(L_n^{\frac{k}{n}}\right)_{\geqslant m}\right\}_\kappa \qquad k \neq ln \tag{19}$$

where k, l are integers and $()_{\geqslant m}$ denotes the projection with respect to powers of Π with the star product, defines a consistent Lax equation only if $m = 0, 1, 2$. Note here that for any element A of the sMm algebra, $A^{\frac{k}{n}} = A^{\frac{1}{n}} \star A^{\frac{1}{n}} \star \cdots \star A^{\frac{1}{n}}$, where the nth root is determined formally in a recursive manner. The projection with $m = 0$ is conventionally denoted as $()_+$ and an equation with the projection $m = 0$ is called a standard super Moyal–Lax representation while for the other projections, the equations are known as non-standard representations.

Let us describe in detail how all of this works in the case of the $N = 1$ supersymmetric KdV hierarchy. Let us consider a fermionic superfield of the form

$$\Phi(x, \theta) = \psi(x) + \theta u(x) \tag{20}$$

and a Lax function which is an element of the sMm algebra of the form

$$L = p^2 + \Pi \star \Phi = p^2 - \Phi \star \Pi + 2\kappa(D\Phi). \tag{21}$$

In this case, we can show in a straightforward manner (all products and projections are with respect to star product) that

$$\left(L^{\frac{3}{2}}\right)_+ = p^3 - \frac{3}{2}\Phi \star p\Pi + 3\kappa(D\Phi) \star p + \frac{3\kappa}{2}\Phi_x \star \Pi - 3\kappa^2(D\Phi_x) \tag{22}$$

where the subscript x stands for a derivative with respect to the space coordinate. It is tedious, but straightforward to check that the super Moyal–Lax equation (in the standard representation)

$$\frac{\partial L}{\partial t} = \frac{2}{\kappa}\left\{L, \left(L^{\frac{3}{2}}\right)_+\right\}_\kappa \tag{23}$$

leads to

$$\frac{\partial \Phi}{\partial t} = (2\kappa\Phi_{xx} + 3\Phi(D\Phi))_x \tag{24}$$

which we recognize as the $N = 1$ supersymmetric KdV equation [2] (the conventional representations of the equation corresponds to $2\kappa = 1$). In fact, the entire supersymmetric KdV hierarchy can be obtained from a super Moyal–Lax equation of the form (up to a multiplicative constant)

$$\frac{\partial L}{\partial t_k} = \left\{L, \left(L^{\frac{2k+1}{2}}\right)_+\right\}_\kappa. \tag{25}$$

Let us next show that the super Moyal–Lax equation of (25) can be derived from an action and can be given the meaning of a Hamiltonian equation. Let us consider a phase space action of the form

$$S = \int \mathrm{d}t\,\left(p \star \dot{x} + \dot{\theta} \star p_\theta - \left(L^{\frac{2k+1}{2}}\right)_+\right). \tag{26}$$

Here the particular ordering of the velocity in the second term reflects our choice of a left derivative [21, 22]. It can now be easily checked that the Euler–Lagrange equations following from this action lead to

$$\begin{aligned}
\dot{x} &= \frac{\partial\big(L^{\frac{2k+1}{2}}\big)_+}{\partial p} = \big\{x, \big(L^{\frac{2k+1}{2}}\big)_+\big\}_\kappa \\
\dot{p} &= -\frac{\partial\big(L^{\frac{2k+1}{2}}\big)_+}{\partial x} = \big\{p, \big(L^{\frac{2k+1}{2}}\big)_+\big\}_\kappa \\
\dot{\theta} &= -\frac{\partial\big(L^{\frac{2k+1}{2}}\big)_+}{\partial p_\theta} = \big\{\theta, \big(L^{\frac{2k+1}{2}}\big)_+\big\}_\kappa \\
\dot{p}_\theta &= -\frac{\partial\big(L^{\frac{2k+1}{2}}\big)_+}{\partial \theta} = \big\{p_\theta, \big(L^{\frac{2k+1}{2}}\big)_+\big\}_\kappa.
\end{aligned} \tag{27}$$

Namely, these are the appropriate Hamiltonian equations for the system with the super Moyal bracket playing the role of the Poisson bracket and $\big(L^{\frac{2k+1}{2}}\big)_+$ representing the Hamiltonian. The dynamical evolution of any other variable can now be obtained in a simple manner and, in particular, we note that

$$\frac{\partial L}{\partial t_k} = \big\{L, \big(L^{\frac{2k+1}{2}}\big)_+\big\}_\kappa. \tag{28}$$

This shows that the super Moyal–Lax equation can indeed be thought of as a Hamiltonian equation with $\big(L^{\frac{2k+1}{2}}\big)_+$ playing the role of the Hamiltonian, much like in bosonic integrable systems [14]. Although we have shown this explicitly for a super Moyal–Lax equation in the standard representation, it is quite clear that this derivation generalizes to a super Moyal–Lax equation with a non-standard representation as well as systems with extended supersymmetry.

The conserved quantities of the system can be obtained in a simple manner. Using the definition of *super-trace* in equation (18), we can write (up to a multiplicative constant)

$$H_{2m+1} = -\frac{1}{(2m+1)\kappa^m}\, s\,\mathrm{Tr}\, L^{\frac{2m+1}{2}} \tag{29}$$

and expressing the Hamiltonians in terms of densities as

$$H_{2m+1} = \int \mathrm{d}x\, \mathrm{d}\theta\, \mathcal{H}_{2m+1}$$

the first few Hamiltonian densities take the forms

$$\begin{aligned}
\mathcal{H}_1 &= \Phi \\
\mathcal{H}_3 &= \tfrac{1}{4}\,\Phi(D\Phi) \\
\mathcal{H}_5 &= \tfrac{1}{4}\big(\kappa\Phi(D\Phi_{xx}) + \Phi(D\Phi)^2\big) \\
\mathcal{H}_7 &= \tfrac{1}{16}\big[4\kappa^2\Phi(D\Phi_{xxxx}) + 2\kappa\big(\Phi_{xx}\Phi_x\Phi + 7\Phi(D\Phi_{xx})(D\Phi) + 4(D\Phi)^2\big) + 5\Phi(D\Phi)^3\big]
\end{aligned} \tag{30}$$

and so on. It is easy to check that these quantities are conserved under the flow of the sKdV hierarchy.

The discussion of the Hamiltonian structures for the sKdV hierarchy can be carried out much along the lines of pseudo-differential operators. Since it is rather technical, we simply give the results here. The first Hamiltonian structure is highly non-local and has the form

$$\mathcal{D}_1 = 8\kappa D^2(2\kappa D^3 + \Phi)^{-1}D^2 \tag{31}$$

while the second structure, corresponding to the superconformal algebra, has the form

$$\mathcal{D}_2 = 2\big(2\kappa D^5 + 3\Phi D^2 + (D\Phi)D + 2(D^2\Phi)\big) \tag{32}$$

so that the sKdV equation, equation (24), can be written as

$$\frac{\partial \Phi}{\partial t} = \mathcal{D}_1 \frac{\delta H_5}{\delta \Phi} = \mathcal{D}_2 \frac{\delta H_3}{\delta \Phi}. \tag{33}$$

From the structure in (32), it is also clear that the non-commutativity parameter, κ, is related to the central charge of the superconformal algebra, much as we had shown earlier [14] that in a bosonic integrable model, it is related to the central charge in the algebra of the second Hamiltonian structure.

Let us note here that in the limit $\kappa \to 0$, equation (24) does lead to the dispersionless limit of the $N = 1$ supersymmetric KdV equation [17], namely,

$$\frac{\partial \Phi}{\partial t} = 3\,(\Phi(D\Phi))_x\,. \tag{34}$$

However, we note from equation (23) that this is obtained in a singular limit which is not very amenable to manipulations. It is for this reason that an alternate Lax representation for the dispersionless equation has proven much more useful [17] and the origin of such an alternate Lax representation remains an open question. It is also worth noting here that, in the limit $\kappa \to 0$, the conserved quantities in equations (29), (30) do reduce to the local conserved quantities of the dispersionless sKdV hierarchy. Furthermore, in this limit, the second Hamiltonian structure in equation (32) reduces to the center-less superconformal algebra, which is known to be a Hamiltonian structure of the dispersionless model. On the other hand, the first Hamiltonian structure in equation (31) vanishes in this limit, which explains why such a structure has not been found within the context of the dispersionless model. Without going into details, we will like to note here that the super Moyal–Lax representation also allows us to construct the non-local charges of the $N = 1$ supersymmetric KdV system, which reduces in the dispersionless limit to one of the two sets of non-local charges found in the literature [17]. The understanding of the non-local charges, within the context of dispersionless supersymmetric systems, therefore, remains an open question.

4. Other examples

In this section, we will discuss briefly the super Moyal–Lax representations for some other integrable models with $N = 1$ supersymmetry.

4.1. Nonstandard KdV

$N = 1$ supersymmetric KdV can also be given a non-standard description as follows. Let us consider the Lax operator

$$L = p + p^{-1} \star \Pi \star \Phi. \tag{35}$$

Then, it is easy to check that (projection with respect to the star product and powers of Π),

$$(L^3)_{\geqslant 1} = p^3 + 6\kappa(D\Phi) \star p - 3\Phi \star p \star \Pi. \tag{36}$$

It follows from this that the super Moyal–Lax equation

$$\frac{\partial L}{\partial t} = \frac{1}{2\kappa}\left\{L, \left(L^3\right)_{\geqslant 1}\right\}_\kappa \tag{37}$$

leads to the $N = 1$ susy KdV equation

$$\frac{\partial \Phi}{\partial t} = \left(4\kappa^2 \Phi_{xx} + 3\Phi(D\Phi)\right)_x. \tag{38}$$

Once again, we see that the dispersionless limit can be obtained in the singular limit, $\kappa \to 0$.

4.2. Supersymmetric two-boson equation

The supersymmetric two-boson equation [5] also has a non-standard super Moyal–Lax representation of the following form. Let us consider the Lax operator of the form

$$L = p - (D\Phi_0) + \Pi^{-1} \star \Phi_1. \tag{39}$$

Here, both Φ_0 and Φ_1 are considered to be fermionic super-fields. Then, it follows that

$$\left(L^2\right)_{\geqslant 1} = p^2 - 2(D\Phi_0) \star p + (2\kappa)^{-1}\Phi_1 \star \Pi \tag{40}$$

and the super Moyal–Lax equation

$$\frac{\partial L}{\partial t} = \left\{L, \left(L^2\right)_{\geqslant 1}\right\}_\kappa \tag{41}$$

leads to the consistent Hamiltonian equations

$$\begin{aligned} \frac{\partial \Phi_0}{\partial t} &= -\left(2\kappa\Phi_{0xx} + (D(D\Phi_0)^2) + 2\Phi_{1x}\right) \\ \frac{\partial \Phi_1}{\partial t} &= \left(2\kappa\Phi_{1x} + 2\Phi_1(D\Phi_0)\right)_x. \end{aligned} \tag{42}$$

These equations are easily seen to reduce to the correct dispersionless system [18] in the limit $\kappa \to 0$. Let us also note here that the supersymmetric KdV hierarchy is embedded in the supersymmetric two boson hierarchy (with $\Phi_0 = 0$).

4.3. Supersymmetric nonlinear Schrödinger equation

It is known that the supersymmetric two-boson equation is related to the supersymmetric nonlinear Schrödinger equation through a field redefinition [5]. Let us next show that this holds even with a Moyal–Lax representation, which clarifies some of the features of the dispersionless limit of this model. Let us define

$$\Phi_0 = 2\kappa(D\ln(DQ)) + (D^{-1}(\overline{Q}Q)) \qquad \Phi_1 = 2\kappa\overline{Q}(DQ). \tag{43}$$

With this redefinition, the Lax operator for the susy two boson hierarchy becomes

$$\begin{aligned} L &= p - (D\Phi_0) + \Pi^{-1} \star \Phi_1 \\ &= p - 2\kappa\frac{(D^3Q)}{(DQ)} - \overline{Q}Q + 2\kappa\Pi^{-1} \star \overline{Q}(DQ) \\ &= (DQ)^{-1} \star \left(p - \overline{Q}Q + 2\kappa(DQ)^{-1} \star \Pi^{-1} \star \overline{Q}\right) \star (DQ) \\ &= G \star \tilde{L} \star G^{-1}. \end{aligned} \tag{44}$$

One says that the Lax operators L and $\tilde{L}$ are related through a gauge transformation G.

It is easy to verify that the new Lax operator $\tilde{L}$ does not lead to any consistent Moyal–Lax equation. However, let us define (in the language of pseudo-differential operators, this will be called a formal adjoint)

$$\mathcal{L} = \tilde{L}^T = -\left(p + \overline{Q}Q - 2\kappa\overline{Q} \star \Pi^{-1} \star (DQ)\right). \tag{45}$$

Then, the super Moyal–Lax equation

$$\frac{\partial \mathcal{L}}{\partial t} = \frac{1}{4\kappa}\left\{\mathcal{L}, \left(\mathcal{L}^2\right)_{\geqslant 1}\right\}_\kappa \tag{46}$$

leads to

$$\begin{aligned} \frac{\partial Q}{\partial t} &= \kappa Q_{xx} + \left(D((DQ)\overline{Q})\right)Q \\ \frac{\partial \overline{Q}}{\partial t} &= -\kappa\overline{Q}_{xx} - \left(D((D\overline{Q})Q)\right)\overline{Q} \end{aligned} \tag{47}$$

which are the supersymmetric nonlinear Schrödinger equations. This shows that a dispersionless limit of this set of equations can be obtained in the singular $\kappa \to 0$ limit (note the powers of κ in the field redefinitions as well), which explains why a direct construction of such a model has not succeeded so far.

4.4. Supersymmetric mKdV equation

We also note here that, if we make the identification $\overline{Q} = Q$ (recall that Q is fermionic), then the Lax operator in equation (45) will define a consistent super Moyal–Lax equation

$$\frac{\partial \mathcal{L}}{\partial t} = \frac{1}{2\kappa}\left\{\mathcal{L}, \left(\mathcal{L}^3\right)_{\geqslant 1}\right\}_\kappa \tag{48}$$

leading to

$$\frac{\partial Q}{\partial t} = 2\kappa Q_{xxx} - 3(D^2(Q(DQ))(DQ). \tag{49}$$

We recognize this to be the $N = 1$ supersymmetric mKdV (modified KdV) equation. In the limit $\kappa \to 0$, this leads to the correct dispersionless limit (be it in a singular manner)

$$\frac{\partial Q}{\partial t} = -3(D^2(Q(DQ))(DQ) \tag{50}$$

whose bosonic limit yields the dispersionless limit of the mKdV equation (which is a higher-order flow of the Riemann hierarchy).

5. Examples of systems with extended supersymmetry

In this section, we will discuss the distinct $N = 2$ supersymmetrizations of the KdV equation as examples of systems with extended supersymmetry. Let us note that the natural setting for a description of the $N = 2$ supersymmetric KdV hierarchies is the $N = 2$ superspace, which is parameterized by two fermionic (Grassmann) coordinates θ_1, θ_2 in addition to the usual bosonic coordinate x (in the notation of section 2, $n = 1$, $N = 2$). In this case, there are two possible covariant derivatives that one can define, namely,

$$D_1 = \partial_{\theta_1} + \theta_1\partial_x \qquad D_2 = \partial_{\theta_2} + \theta_2\partial_x. \tag{51}$$

These covariant derivatives satisfy the algebraic relations

$$D_1^2 = D_2^2 = \partial_x \qquad D_1D_2 = -D_2D_1. \tag{52}$$

Correspondingly, on the phase space manifold of such a system, we can define two variables

$$\Pi_1 = -(p_{\theta_1} + \theta_1 p) \qquad \Pi_2 = -(p_{\theta_2} + \theta_2 p) \tag{53}$$

which would satisfy (see equation (2))

$$\Pi_1 \star \Pi_1 = -2\kappa p = \Pi_2 \star \Pi_2 \qquad \Pi_1 \star \Pi_2 = -\Pi_2 \star \Pi_1. \tag{54}$$

Furthermore, through the graded Moyal bracket, they will lead to covariant derivatives acting on any superfield on this space.

With these, let us consider a bosonic superfield Ψ on this graded manifold depending on x, θ_1, θ_2. This $N = 2$ superfield can, of course, be decomposed and written as a sum of two $N = 1$ superfields, but let us continue our discussion with Ψ. It is known that there are only three nontrivial $N = 2$ supersymmetrizations of the KdV hierarchy which are integrable (corresponding to a parameter $a = 1, 4, -2$). Let us consider the three cases separately.

The Lax operator

$$L = p + \Pi_1^{-1} \star \Pi_2 \star \Psi \tag{55}$$

leads through the super Moyal–Lax equation

$$\frac{\partial L}{\partial t} = \left\{L, \left(L^3\right)_{\geqslant 1}\right\}_\kappa \tag{56}$$

to the equation

$$\frac{\partial \Psi}{\partial t} = \left(-4\kappa^2\Psi_{xx} - 6\kappa(D_1 D_2\Psi)\Psi + \Psi^3\right)_x \tag{57}$$

which we recognize to be the $N = 2$ supersymmetrization of the KdV equation corresponding to $a = 1$ [6].

On the other hand, the Lax operator

$$L = -(\Pi_1 \star \Pi_2 + \Psi)^2 \tag{58}$$

leads, through the standard super Moyal–Lax equation,

$$\frac{\partial L}{\partial t} = -\frac{1}{2\kappa}\left\{L, \left(L^{\frac{3}{2}}\right)_+\right\}_\kappa \tag{59}$$

to the equation

$$\frac{\partial \Psi}{\partial t} = \left(-4\kappa^4\Psi_{xx} + 3\kappa^2(D_1\Psi)(D_2\Psi) + 6\kappa^2(D_1 D_2\Psi)\Psi + \Psi^3\right)_x \tag{60}$$

which is the $N = 2$ supersymmetrization of the KdV equation corresponding to $a = 4$ [3].

Finally, we note that if we take the Lax operator to be the $()_{\geqslant 1}$ projection of that in equation (58), namely,

$$\mathcal{L} = (L)_{\geqslant 1} \tag{61}$$

then, the standard super Moyal–Lax equation

$$\frac{\partial \mathcal{L}}{\partial t} = \frac{1}{\kappa}\left\{\mathcal{L}, \left(\mathcal{L}^{\frac{3}{2}}\right)_+\right\}_\kappa \tag{62}$$

leads to the equation

$$\frac{\partial \Psi}{\partial t} = \left(8\kappa^4\Psi_{xx} - 6\kappa^2(D_1\Psi)(D_2\Psi) + \Psi^3\right)_x \tag{63}$$

which is the $N = 2$ supersymmetrization of the KdV equation corresponding to the parameter $a = -2$ [3].

Thus, we see that all three of the $N = 2$ supersymmetrizations of the KdV equation can be given a super Moyal–Lax representation. We can carry over the arguments of section 3 and show that all the three super Moyal–Lax equations can be given the meaning of Hamiltonian equations and can be derived from suitable actions. Furthermore, the dispersionless limits of these models can be obtained by taking the $\kappa \to 0$ limit. Surprisingly, all three models, in this limit, yield

$$\frac{\partial \Psi}{\partial t} = \left(\Psi^3\right)_x \tag{64}$$

whose bosonic limit is

$$\begin{aligned} \frac{\partial \omega}{\partial t} &= \left(\omega^3\right)_x \\ \frac{\partial u}{\partial t} &= 3\,(u\omega)_x \end{aligned} \tag{65}$$

where we have identified $\Psi = \omega + \theta_1\psi_1 + \theta_2\psi_2 + \theta_2\theta_1 u$ and this equation contains the dispersionless mKdV equation (in the limit $u = 0$). Thus, equation (64), can be thought of as a trivial supersymmetrization of the dispersionless mKdV equation (in the sense that it does not contain supersymmetric covariant derivatives).

We can, in fact, give a Lax description for this supersymmetrization of the mKdV equation [23] in the following way. First, let us consider a rotated basis and define two covariant derivatives as

$$D_1 = \partial_{\theta_1} - \tfrac{1}{2}\theta_2\partial_x \qquad D_2 = \partial_{\theta_2} - \tfrac{1}{2}\theta_1\partial_x. \tag{66}$$

Unlike the conventional ones, these covariant derivatives satisfy

$$D_1^2 = 0 = D_2^2 \qquad D_1D_2 + D_2D_1 = -\partial_x. \tag{67}$$

In this case, we can define

$$\Pi_1 = -p_{\theta_1} + \tfrac{1}{2}\theta_2 p \qquad \Pi_2 = -p_{\theta_2} + \tfrac{1}{2}\theta_1 p \tag{68}$$

such that, with the star product defined in equation (2), we have

$$\Pi_1 \star \Pi_1 = 0 = \Pi_2 \star \Pi_2 \qquad \{\Pi_1, \Pi_2\}_\kappa = p. \tag{69}$$

Furthermore, it can be checked that $\Pi_{1,2}$, through the super Moyal bracket, generate appropriate covariant derivatives in the rotated basis.

With these operators, let us next define

$$L = p^2 + \Psi \star p + (D_2\Psi) \star \Pi_1. \tag{70}$$

Then, it is easy to check that the super Moyal–Lax equation

$$\frac{\partial L}{\partial t} = -8\{L, (L^{\frac{3}{2}})_{\geqslant 1}\}_\kappa \tag{71}$$

leads to

$$\frac{\partial \Psi}{\partial t} = \left(- 8\kappa^2\Psi_{xx} + 12\kappa(D_1\Psi)(D_2\Psi) + \Psi^3\right)_x. \tag{72}$$

This equation can be compared with equation (63) (recall, however, that the covariant derivatives in the two equations correspond to different basis). In fact, we note here that the dispersionless limit of this equation,

$$\frac{\partial \Psi}{\partial t} = (\Psi^3)_x \tag{73}$$

can be described by a Lax equation of the following form. Let us consider

$$L = p^2 + \Psi p. \tag{74}$$

Then, the ordinary Poisson bracket equation,

$$\frac{\partial L}{\partial t} = -8\{L, (L^{\frac{3}{2}})_{\geqslant 1}\} \tag{75}$$

leads to equation (73). It is not obvious that the Lax operator (70) reduces exactly to that in equation (74). However, that this is true can be seen in the following manner. The linear equation associated with the Lax operator in equation (70) has the form

$$L \star \psi = \lambda\psi \tag{76}$$

where λ represents the spectral parameter. Both the eigenfunction and the spectral parameter are functions of κ, the deformation parameter. Therefore, they can be expanded in a power series in κ, as, say

$$\psi = \psi_0 + \kappa\psi_1 + \kappa^2\psi_2 + \cdots. \tag{77}$$

It can be determined easily that ψ_0, the component which survives in the dispersionless limit ($\kappa \to 0$), has the form $\psi_0 = \Pi_1\phi$. As a result of this in the dispersionless limit, the last term of the Lax operator (70) drops out in the linear equation (76) (recall that in the dispersionless limit, $\kappa \to 0$, we have $\Pi_1^2 = 0$) so that the Lax operator in (74) truly represents the reduction of equation (70) in the dispersionless limit.

Let us also note here that equation (73) can be checked to have at least three Hamiltonian structures of the forms

$$\mathcal{D}_1 = \partial \qquad \mathcal{D}_2 = \partial\Psi + \Psi\partial \qquad \mathcal{D}_3 = \partial\Psi\partial^{-1}\Psi\partial \tag{78}$$

each of which can be checked to satisfy the Jacobi identity.

6. Conclusion

We have generalized our earlier discussion of Moyal–Lax representation for bosonic integrable systems [14] to supersymmetric ones, with simple as well as extended supersymmetries. We have derived various properties of the supersymmetric star product. Within the context of the $N = 1$ supersymmetric KdV equation, we have shown how the parameter of deformation, in such systems, is related to the central charge of the second Hamiltonian structure. We have also shown how the super Moyal–Lax equation can be interpreted as a Hamiltonian equation and can be derived from an action, much like in the bosonic case. The conserved charges as well as the first two Hamiltonian structures are constructed. We show how one can take the dispersionless limit of this model within the super Moyal–Lax representation, the limit being singular. (It is for this reason that the standard construction of a Lax description for such dispersionless systems fails and one needs an alternate description [17, 18].) The conserved quantities as well as the Hamiltonian structures also reduce to the corresponding quantities of the dispersionless models. This clarifies why the construction of the first Hamiltonian structure for the dispersionless sKdV system had failed so far. We have also briefly discussed the super Moyal–Lax representations for the $N = 1$ supersymmetric two boson equation, nonlinear Schrödinger equation as well as the modified KdV equation. We have also discussed the super Moyal–Lax representations for the various $N = 2$ supersymmetrizations of the KdV equation. The dispersionless limits of these systems and their properties are also discussed.

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